

# Solvability of Dirac type equations

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## Abstract

This paper develops a weighted  $L^2$ -method for the (half) Dirac equation. For Dirac bundles over closed Riemann surfaces, we give a sufficient condition for the solvability of the (half) Dirac equation in terms of a curvature integral. Applying this to the Dolbeault-Dirac operator, we establish an automatic transversality criteria for holomorphic curves in Kähler manifolds. On compact Riemannian manifolds, we give a new perspective on some well-known results about the first eigenvalue of the Dirac operator, and improve the estimates when the Dirac bundle has a  $\mathbb{Z}_2$ -grading. On Riemannian manifolds with cylindrical ends, we obtain solvability in the  $L^2$ -space with suitable exponential weights while allowing mild negativity of the curvature.

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## 1 Introduction

In many geometric problems, it is important to determine the solvability of the linear equation

$$Du = f \tag{1}$$

where  $D$  is the Dirac operator on some Dirac bundle. For example, the fundamental Dirac operator on spin manifolds ([1]), the Dolbeault-Dirac operator

in Kähler geometry, and the twisted Dirac operator in the normal bundle of instantons (associative submanifolds) in  $G_2$  manifolds ([22]). In general, it is not easy to know when (1) is solvable. For the Dirac operator on spin manifolds, a sufficient condition was given by the positivity of the scalar curvature, dating back to a theorem of Lichnerowicz. However, the positive scalar curvature condition is not always necessary, as the Dirac operator on spin manifolds has the remarkable conformal covariance property ([12]), and a conformal change of metric could make the scalar curvature negative somewhere.

In this paper, starting with the Bochner formula, we establish weighted  $L^2$ -estimates and existence theorems for the Dirac equation, just as Hörmander's weighted  $L^2$ -method for the  $\bar{\partial}$ -equation ([15], [16]). In applications of the  $L^2$ -method, it is very important to construct good weight functions from geometric conditions (e.g. [4], [24]~[26]). Sometimes one can gain "extra positivity" in suitable weighted  $L^2$ -spaces to establish vanishing theorems.

Let  $\lambda_{\mathbb{S}}$  be the function on  $M$  defined in (19), which pointwisely is the first eigenvalue of some curvature operator. For Dirac equations on 2-dimensional Riemannian manifolds, taking  $n = 2$  in Proposition 2.7, we have

**Theorem 1.1** *Let  $\mathbb{S}$  be a Dirac bundle over a 2-dimensional Riemannian manifold  $(M, g)$  and  $D$  be the Dirac operator. Suppose there exists a  $C^2$  function  $\varphi : M \rightarrow \mathbb{R}$  such that  $\Delta\varphi + 2\lambda_{\mathbb{S}} \geq 0$  on  $M$ . If a section  $f$  of  $\mathbb{S}$  satisfies  $\int_M \frac{|f|^2}{\Delta\varphi + 2\lambda_{\mathbb{S}}} e^{-\varphi} < \infty$ , then there exists a section  $u$  of  $\mathbb{S}$  such that*

$$Du = f, \text{ and } \int_M |u|^2 e^{-\varphi} \leq \int_M \frac{|f|^2}{\Delta\varphi + 2\lambda_{\mathbb{S}}} e^{-\varphi}. \quad (2)$$

Our theorem 1.1 leads to the following solvability criterion of the half Dirac equation on  $\mathbb{Z}_2$ -graded Dirac bundles (see Section 2.1 for definitions).

**Corollary 1.2** *Let  $\mathbb{S}$  be a  $\mathbb{Z}_2$ -graded Dirac bundle over a closed 2-dimensional Riemannian manifold  $M$  and  $D^{\pm}$  be the half Dirac operators, then*

$$\lambda_{\min}(D^{\pm}D^{\mp}) \geq \frac{2}{\text{Vol}(M)} \int_M \lambda_{\mathbb{S}^{\mp}}, \quad (3)$$

where  $\lambda_{\min}(D^{\pm}D^{\mp})$  is the first eigenvalue of  $D^{\pm}D^{\mp}$ . Consequently, if

$$\int_M \lambda_{\mathbb{S}^{\mp}} > 0, \quad (4)$$

then for any  $f \in L^2(M, \mathbb{S}^{\mp})$ , there exists a section  $u \in L^2(M, \mathbb{S}^{\pm})$  such that  $D^{\pm}u = f$ .

Applying Corollary 1.2 to the Dolbeault-Dirac operator, we obtain

**Corollary 1.3** *Let  $E$  be a holomorphic vector bundle over a closed Riemann surface  $M$  and  $\bar{\partial} : E \rightarrow \wedge^{0,1}(E)$  be the Cauchy-Riemann operator. For any Hermitian metric on  $E$ , we have the following estimates:*

$$\lambda_{\min}(\bar{\partial}^* \bar{\partial}) \geq \frac{-1}{\text{Vol}(M)} \int_M \Theta_E, \quad (5)$$

$$\lambda_{\min}(\bar{\partial} \bar{\partial}^*) \geq \frac{1}{\text{Vol}(M)} \left( \int_M \theta_E + 2\pi \chi(M) \right). \quad (6)$$

In particular, if  $E$  is a line bundle, we have

$$\lambda_{\min}(\bar{\partial}^* \bar{\partial}) \geq \frac{-2\pi c_1(E)}{\text{Vol}(M)}, \quad \lambda_{\min}(\bar{\partial} \bar{\partial}^*) \geq \frac{2\pi (c_1(E) + \chi(M))}{\text{Vol}(M)}, \quad (7)$$

where  $\theta_E$  and  $\Theta_E$  are defined by (22) and (23) respectively,  $c_1(E)$  is the first Chern number of  $E$ , and  $\chi(M)$  is the Euler number of  $M$ .

The above Corollary generalizes Bär's first eigenvalue estimate ([2]) for the classic Dirac operator on closed Riemann surfaces, where  $E = K_M^{\frac{1}{2}}$  and the Riemann surface is of genus 0.

Corollary 1.3 has a potential extension to real linear Cauchy-Riemann operators on Riemann surfaces, which in turn has applications to the “automatic transversality” criteria of  $J$ -holomorphic curves in almost complex manifolds  $X$ . We recall the definitions below.

Given a Riemann surface  $(M, j)$ , where  $j$  is its complex structure, a  $J$ -holomorphic curve  $u : (M, j) \rightarrow (X, J)$  is a solution of the nonlinear Cauchy-Riemann equation  $\bar{\partial}_J u = 0$ , where  $\bar{\partial}_J(u) = \frac{1}{2}(du + J \circ du \circ j)$ . For a  $J$ -holomorphic curve  $u$ , the *linearized operator*

$$D_u : W^{1,p}(M, u^*TX) \rightarrow L^p(M, \wedge^{0,1}(u^*TX))$$

( $p \geq 2$ ) of the nonlinear Cauchy-Riemann operator  $\bar{\partial}_J$  is of the form

$$D_u = \bar{\partial} + A,$$

where  $\bar{\partial}$  is the Cauchy-Riemann operator on the complex vector bundle  $u^*TX$  and  $A : W^{1,p}(M, u^*TX) \rightarrow L^p(M, \wedge^{0,1}(u^*TX))$  is an anti-complex linear homomorphism determined by the Nijenhuis tensor of  $J$ .  $A = 0$  if  $J$  is integrable.  $\bar{\partial} + A$  is a *real linear Cauchy-Riemann operator* (c.f. Appendix C [23]).

**Definition 1.4** *A  $J$ -holomorphic curve  $u$  is called transversal (or Fredholm regular), if  $D_u$  is surjective. A  $J$ -holomorphic curve  $u$  has automatic transversality, if  $u$  is transversal regardless of whether  $J$  is generic or not.*

Note for symplectic manifolds  $(X, \omega)$  of dimension 4 with compatible almost complex structures  $J$ , for immersed  $J$ -holomorphic curves  $u$ , taking  $E = N_u$  (the normal bundle of  $u$  in  $X$ ), the estimate (7) is related to the well-known

Chern number condition  $c_1(E) > -\chi(M)$  to ensure automatic transversality, which was first mentioned in [8] and later established in [14], [28], [23], with nice applications to the moduli space theory of  $J$ -holomorphic curves.

If the almost complex structure  $J$  is integrable, our Corollary 1.3 provides a criterion for automatic transversality of holomorphic curves in Kähler manifolds with real dimensional  $\geq 4$ .

**Corollary 1.5** *Let  $u : (M, j) \rightarrow (X, \omega, J)$  be a holomorphic curve in a Kähler manifold  $X$  equipped with complex structure  $J$  and Kähler form  $\omega$ . Let  $E = u^*TX$  be the vector bundle over  $M$  with induced holomorphic structure and metric. Then  $u$  has automatic transversality, if*

$$\int_M \theta_E > -2\pi\chi(M), \quad (8)$$

where  $\theta_E$  is defined in (22).

We leave the general case to our future work [18].

For spinor bundles over closed spin manifolds of dimension  $\geq 3$ , Hijazi obtained ([11]) a lower bound of the spectrum of the fundamental Dirac operator in terms of the first eigenvalue of the Yamabe operator, with the help of the conformal covariance of the Dirac operator and the transformation law of the scalar curvature under a conformal change of the given metric. By introducing a new connection, Bär ([2]) generalized this result, using a delicate technique of completing square, to any Dirac bundle over a closed Riemannian manifold.

As an application of our weighted  $L^2$ -estimates, we construct weight functions by solving certain partial differential equations and give a new proof of Bär's results on the first eigenvalue of the Dirac operator (Theorem 3.5).

For Dirac bundles with  $\mathbb{Z}_2$ -gradings, we establish similar results for the (half) Dirac operator  $D^\pm$  in Section 4, as this is important for many geometric applications. Using the  $\mathbb{Z}_2$ -grading (which always exists on Dirac bundles over even dimensional manifolds) we improve Bär's ([2] Theorem 3) first eigenvalue estimates of  $D$  on even dimensional manifolds.

In gauge theory, it occurs often that  $M$  are Riemannian manifolds with *cylindrical ends* (Definition 5.1), and exponential weights on the ends are frequently used to set up the moduli spaces (e.g. [5], [27]). On such manifolds we have the following existence theorem without assuming  $\lambda_{\mathbb{S}}$  is positive everywhere.

**Theorem 1.6** *Suppose  $\mathbb{S}$  is a Dirac bundle over a Riemannian manifold  $(M, g)$  with cylindrical ends,  $\dim M \geq 3$ . Suppose for some compact subset  $K$ ,  $M \setminus K$  is contained in the cylindrical ends, and there exist a constant  $\alpha > 0$  such that*

$$\lambda_{\mathbb{S}} \geq \alpha \text{ on } M \setminus K. \quad (9)$$

*Then there exists a constant  $\beta > 0$  with the following significance: when*

$$\lambda_{\mathbb{S}} \geq -\beta \text{ on } K, \quad (10)$$

*we have*

1. for any section  $f \in L^2_\delta(M, \mathbb{S})$ , there exists a section  $u \in W^{1,2}_\delta(M, \mathbb{S})$  such that

$$Du = f \text{ and } \|u\|_{W^{1,2}_\delta(M, \mathbb{S})} \leq C \|f\|_{L^2_\delta(M, \mathbb{S})}; \quad (11)$$

2. for any section  $f \in L^p(M, \mathbb{S})$  ( $p > 1$ ), there exists a section  $u \in W^{1,p}(M, \mathbb{S})$  such that

$$Du = f \text{ and } \|u\|_{W^{1,p}(M, \mathbb{S})} \leq C_p \|f\|_{L^p(M, \mathbb{S})}. \quad (12)$$

Here  $\delta_1, \dots, \delta_m \geq 0$  are sufficiently small,  $\delta = (\delta_1, \dots, \delta_m)$  is the exponential weight (defined in (52)) for the cylindrical ends, and the constants  $C$  and  $C_p$  are independent on  $f$ .

Note that the  $L^2$ -solvability of the Dirac equation on a complete Riemannian manifold was established in Theorem 2.11 of [7], with the assumption that  $\lambda_{\mathbb{S}} \geq \alpha > 0$ . Our theorem allows mild negativity of  $\lambda_{\mathbb{S}}$  on a compact domain  $K \subset M$  (as in (10)), when  $M$  is a Riemannian manifold with cylindrical ends. Some recent results of the Dirac equation on non-compact Riemannian manifolds can be found in [3], and [9]~[10].

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## 2 Weighted $L^2$ -estimates for Dirac operators

### 2.1 Dirac bundles and Dirac operators

In this section, we recall some basic facts of the Dirac operator and set up the notations. Let  $(M, g)$  be a smooth  $n$ -dimensional Riemannian manifold and  $Cl(M, g) \rightarrow M$  be the corresponding *Clifford bundle*. Let  $\mathbb{S} \rightarrow M$  be a bundle of left  $Cl(M, g)$ -modules endowed with a Riemannian metric  $\langle \cdot, \cdot \rangle$  and a Riemannian connection  $\nabla$  such that at any  $x \in M$ , for any unit vector  $e \in T_x M$  and any  $s, s' \in \mathbb{S}_x$ ,

$$\langle e \cdot s, e \cdot s' \rangle = \langle s, s' \rangle \quad (13)$$

where the  $\cdot$  is the Clifford multiplication. Furthermore, for any smooth vector field  $V$  on  $M$  and smooth section  $s$  of  $\mathbb{S}$ ,

$$\nabla(V \cdot s) = (\nabla V) \cdot s + V \cdot (\nabla s), \quad (14)$$

where on the right hand side,  $\nabla$  are the covariant derivatives of the Levi-Civita connection on  $M$  and a connection on  $\mathbb{S}$  for the first and second terms respectively.

**Definition 2.1** A bundle  $(\mathbb{S}, \langle \cdot, \cdot \rangle, \nabla)$  of  $Cl(M, g)$ -modules satisfying (13), (14) is called a *Dirac bundle* over  $M$  (Definition 5.2 [19]) and has a canonically associated Dirac operator  $D$  such that for any section  $s$  of  $\mathbb{S}$ ,

$$Ds = \sum_{i=1}^n e_i \cdot \nabla_{e_i} s, \quad (15)$$

where  $\{e_i\}_{i=1}^n$  is any orthonormal basis of  $T_x M$  for  $x$  on  $M$ .

On the Dirac bundle  $\mathbb{S}$  we can define the *canonical section*  $\mathfrak{R}$  of  $\text{End}(\mathbb{S})$ , such that for any smooth section  $s$  of  $\mathbb{S}$ ,

$$\mathfrak{R}(s) = \frac{1}{2} \sum_{i,j=1}^n e_i \cdot e_j \cdot R_{e_i, e_j}(s), \quad (16)$$

where  $R_{V,W}$  is the curvature transform on  $\mathbb{S}$ . Then we have the Bochner formula (c.f. Theorem 8.2 in Chapter II [19])

$$D^2 = \nabla^* \nabla + \mathfrak{R}. \quad (17)$$

Especially, when  $\mathbb{S}$  is the *spinor bundle* over a spin manifold  $M$ , Lichnerowicz's theorem says

$$\mathfrak{R} = \frac{1}{4} R \cdot \text{Id}_{\mathbb{S}}, \quad (18)$$

where  $R$  is the scalar curvature of  $(M, g)$ , and  $\text{Id}_{\mathbb{S}}$  is the identity map on  $\mathbb{S}$ .

For later applications, for any  $x$  on  $M$  and  $\mathfrak{R}(x) \in \text{End}(\mathbb{S}_x)$ , we let the function

$$\lambda_{\mathbb{S}}(x) = \text{the smallest eigenvalue of } \mathfrak{R}(x), \quad (19)$$

where  $\mathfrak{R}(x)$  is defined in (16). Since  $\mathfrak{R}(x)$  is differentiable with respect to  $x$  on  $M$ , using the definition of eigenvalues by the Rayleigh quotient, it is not hard to see  $\lambda_{\mathbb{S}}(x)$  is a *Lipschitz* function on  $M$ .

In several important applications (see Section 4), one needs to consider the  $\mathbb{Z}_2$ -graded Dirac bundle, i.e.  $\mathbb{S}$  has a parallel decomposition  $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$  so that  $Cl^i(M) \cdot \mathbb{S}^j \subseteq \mathbb{S}^{ij}$  for all  $i, j \in \mathbb{Z}_2 \simeq \{+, -\}$ , where  $ij$  is the sign of the product of the signs  $i$  and  $j$ , and  $Cl^i(M)$  ( $i = +, -$ ) are the even and odd parts of  $Cl(M)$ . The Dirac operator  $D$  splits accordingly

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}, \quad (20)$$

where  $D^{\pm} : \Gamma(\mathbb{S}^{\pm}) \rightarrow \Gamma(\mathbb{S}^{\mp})$ , and  $D^+$  and  $D^-$  are adjoint of each other. Note that for Dirac bundles over *even* dimensional Riemannian manifolds, there always exists a natural  $\mathbb{Z}_2$ -grading (c.f. Section 6 [19]). By (16), we see the curvature operator  $\mathfrak{R}(\cdot)$  splits as well:

$$\mathfrak{R}(\cdot) = \begin{bmatrix} \mathfrak{R}^+(\cdot) & 0 \\ 0 & \mathfrak{R}^-(\cdot) \end{bmatrix}, \quad (21)$$

where  $\mathfrak{R}^+(\cdot) \in \text{End}(\mathbb{S}^+, \cdot)$ , and  $\mathfrak{R}^-(\cdot) \in \text{End}(\mathbb{S}^-, \cdot)$ .

Let  $(M, g)$  be a Riemann surface endowed with a metric and  $(E, h)$  be a Hermitian vector bundle over  $M$ . We denote the twisted  $\text{spin}^c$  bundle by

$$\mathbb{S} = \wedge^{0,*}(M) \otimes E$$

whose Dirac operator is given by

$$D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*).$$

There is a natural  $\mathbb{Z}_2$ -graded structure given by  $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$  where

$$\mathbb{S}^+ = E, \quad \mathbb{S}^- = \wedge^{0,1}(E).$$

It is easy to see that

$$D^+ = \sqrt{2}\bar{\partial}, \quad D^- = \sqrt{2}\bar{\partial}^*.$$

We introduce the following functions

$$\theta_E(x) = \text{the smallest eigenvalue of } \sqrt{-1}\Lambda R^E(x), \quad (22)$$

$$\Theta_E(x) = \text{the biggest eigenvalue of } \sqrt{-1}\Lambda R^E(x), \quad (23)$$

for  $x$  on  $M$ , where  $R^E$  is the curvature of the Chern connection  $\nabla$  of  $(E, h)$ , and  $\Lambda$  is the dual Lefschetz operator of  $(M, g)$ . In this case, we can express  $\mathfrak{R}^\pm$  as follows.

**Lemma 2.2**

$$\mathfrak{R}^- = \sqrt{-1}\Lambda R^E + K, \quad \mathfrak{R}^+ = -\sqrt{-1}\Lambda R^E, \quad (24)$$

where  $K$  is the Gaussian curvature of  $(M, g)$ .

**Proof.** We can verify (24) by a tedious computation according to the definition of  $\mathfrak{R}^\pm$ . We give an alternative proof by using the  $\bar{\partial}$ -Bochner formula. We only prove the first one, the second formula can be proved in the same way. From the  $\bar{\partial}$ -Bochner formula, it follows that on  $\mathbb{S}^-$  we have

$$D^2 = -2\nabla^{\bar{1}}\nabla_{\bar{1}} + 2\sqrt{-1}\Lambda R^E + 2K.$$

Here we have used local coordinate such that  $g_{1\bar{1}} = 1$  at a given point. Since

$$\nabla^*\nabla = -\nabla^{\bar{1}}\nabla_{\bar{1}} - \nabla^1\nabla_1 = -2\nabla^{\bar{1}}\nabla_{\bar{1}} + [\nabla_1, \nabla_{\bar{1}}],$$

we get

$$D^2 = \nabla^*\nabla + \sqrt{-1}\Lambda R^E + K$$

which, comparing with the Bochner formula (17) for Dirac operators, implies the first formula. ■

## 2.2 Weighted $L^2$ -estimates and existence results

Let  $\mathbb{S}$  be a Dirac bundle over a smooth Riemannian manifold  $(M, g)$  and  $D$  be the Dirac operator. For any smooth sections  $s \in \Gamma(M, \mathbb{S})$  with compact support, by (17) and integration by parts, we have

$$\int_M |Ds|^2 = \int_M \left( |\nabla s|^2 + \langle s, \mathfrak{R}s \rangle \right). \quad (25)$$

**Definition 2.3** (Weighted  $L^2$ -space) *Let  $\varphi : M \rightarrow \mathbb{R}$  be a  $C^2$  function. For any sections  $s$  and  $s'$  of  $\mathbb{S}$ , let the weighted inner product of  $s$  and  $s'$  be*

$$(s, s')_\varphi = \int_M \langle s, s' \rangle e^{-\varphi}.$$

where  $\varphi : M \rightarrow \mathbb{R}$  is a  $C^2$  function. Let  $\|s\|_\varphi = \sqrt{(s, s)_\varphi}$  and denote by  $L^2_\varphi(M, \mathbb{S})$  be the space of sections  $s$  of  $\mathbb{S}$  such that  $\|s\|_\varphi < \infty$ . We will drop the subscript  $\varphi$  when  $\varphi = 0$ .

For the Dirac operator  $D : L^2_\varphi(M, \mathbb{S}) \rightarrow L^2_\varphi(M, \mathbb{S})$ , let  $D_\varphi^*$  be its formal adjoint with respect to the measure  $e^{-\varphi} d\text{vol}_g$ . For  $D_\varphi^*$ , we have the following identity which is immediate from definitions.

**Lemma 2.4** *For any smooth section  $s$  of  $\mathbb{S}$ , we have*

$$D_\varphi^* s = e^\varphi D(e^{-\varphi} s) = -\nabla \varphi \cdot s + Ds. \quad (26)$$

We will derive a weighted version of (25) by the technique in [17]. Let  $\Delta$  be the Laplace-Beltrami operator on  $(M, g)$ .

**Proposition 2.5** *For any smooth section  $s$  of  $\mathbb{S}$  with compact support and any  $C^2$  function  $\varphi : M \rightarrow \mathbb{R}$ , we have*

$$\begin{aligned} & \frac{n-1}{n} \int_M |D_\varphi^* s|^2 e^{-\varphi} + \frac{n-2}{n} \text{Re} \int_M \langle \nabla \varphi \cdot s, D_\varphi^* s \rangle e^{-\varphi} \\ & \geq \int_M \left[ \frac{1}{2} \Delta \varphi - \left( \frac{1}{2} - \frac{1}{n} \right) |\nabla \varphi|^2 + \lambda_{\mathbb{S}} \right] |s|^2 e^{-\varphi}. \end{aligned} \quad (27)$$

Let  $\varepsilon > 0$  be a constant, we have

$$\int_M |D_\varphi^* s|^2 e^{-\varphi} \geq C \int_M \left[ \Delta \varphi - \left( 1 - \frac{2}{n} \right) \left( 1 + \frac{1}{\varepsilon} \right) |\nabla \varphi|^2 + 2\lambda_{\mathbb{S}} \right] |s|^2 e^{-\varphi}. \quad (28)$$

where  $C = \frac{n}{2(n-1)+(n-2)\varepsilon}$ . Especially, when  $n = 2$  we have

$$\int_M |D_\varphi^* s|^2 e^{-\varphi} \geq \int_M (\Delta \varphi + 2\lambda_{\mathbb{S}}) |s|^2 e^{-\varphi}. \quad (29)$$



**Proof.** Set  $\sigma := e^{-\frac{\varphi}{2}}s$ , then we know by (26)

$$\begin{aligned}
\int_M |D_\varphi^* s|^2 e^{-\varphi} &= \int_M |e^\varphi D(e^{-\varphi} s)|^2 e^{-\varphi} \\
&= \int_M \left| D\sigma - \frac{1}{2} \nabla \varphi \cdot \sigma \right|^2 \\
&= \int_M \left( |D\sigma|^2 + \frac{1}{4} |\nabla \varphi \cdot \sigma|^2 - \operatorname{Re} \langle \nabla \varphi \cdot \sigma, D\sigma \rangle \right) \\
&= \int_M \left( |\nabla \sigma|^2 + \langle \sigma, \Re \sigma \rangle + \frac{1}{4} |\nabla \varphi|^2 |\sigma|^2 - \operatorname{Re} \langle \nabla \varphi \cdot \sigma, D\sigma \rangle \right), \tag{30}
\end{aligned}$$

where in the last identity we have used (25).

We first rewrite  $\int_M |\nabla \sigma|^2$  as follows

$$\begin{aligned}
\int_M |\nabla \sigma|^2 &= \int_M \left| \nabla s - \frac{1}{2} d\varphi \otimes s \right|^2 e^{-\varphi} \\
&= \int_M \left( |\nabla s|^2 + \frac{1}{4} |\nabla \varphi|^2 |s|^2 \right) e^{-\varphi} - \operatorname{Re} \int_M \langle \nabla s, d\varphi \otimes s \rangle e^{-\varphi} \tag{31}
\end{aligned}$$

For the last term, we have

$$\begin{aligned}
-\operatorname{Re} \int_M \langle \nabla s, d\varphi \otimes s \rangle e^{-\varphi} &= -\operatorname{Re} \sum_{i=1}^n \int_M \langle \nabla_{e_i} s, d\varphi(e_i) s \rangle e^{-\varphi} \\
&= \frac{1}{2} \operatorname{Re} \int_M \nabla_{\nabla(e^{-\varphi})} |s|^2 \\
&= \frac{1}{2} \int_M \left[ \operatorname{div} (|s|^2 \nabla(e^{-\varphi})) - |s|^2 \Delta(e^{-\varphi}) \right] \\
&= \frac{1}{2} \int_M (\Delta \varphi - |\nabla \varphi|^2) |s|^2 e^{-\varphi}, \tag{32}
\end{aligned}$$

where in the third line we have used  $Vf = \operatorname{div}(fV) - f \operatorname{div} V$  for the vector field  $V = \nabla(e^{-\varphi})$  and function  $f = |s|^2$ .

From (31) and (32), we obtain

$$\begin{aligned}
&\int_M |\nabla \sigma|^2 \\
&= \int_M \left( |\nabla s|^2 + \frac{1}{4} |\nabla \varphi|^2 |s|^2 \right) e^{-\varphi} + \frac{1}{2} \int_M (\Delta \varphi - |\nabla \varphi|^2) |s|^2 e^{-\varphi} \\
&= \int_M \left[ |\nabla s|^2 + \left( \frac{1}{2} \Delta \varphi - \frac{1}{4} |\nabla \varphi|^2 \right) |s|^2 \right] e^{-\varphi}. \tag{33}
\end{aligned}$$

For the term  $\operatorname{Re} \langle \nabla \varphi \cdot \sigma, D\sigma \rangle$ , we have

$$\begin{aligned}
\operatorname{Re} \langle \nabla \varphi \cdot \sigma, D\sigma \rangle &= \operatorname{Re} \left\langle e^{-\frac{\varphi}{2}} \nabla \varphi \cdot s, D \left( e^{-\frac{\varphi}{2}} s \right) \right\rangle \\
&= \operatorname{Re} \left\langle \nabla \varphi \cdot s, e^{\frac{\varphi}{2}} D \left( e^{-\frac{\varphi}{2}} s \right) \right\rangle e^{-\varphi} \\
&= \operatorname{Re} \left\langle \nabla \varphi \cdot s, D_{\frac{\varphi}{2}}^* (s) \right\rangle e^{-\varphi} \\
&= \operatorname{Re} \left\langle \nabla \varphi \cdot s, D_{\varphi}^* (s) + \frac{1}{2} \nabla \varphi \cdot s \right\rangle e^{-\varphi}, \tag{34}
\end{aligned}$$

where in the fourth identity we have used Lemma 2.4.

By the Cauchy-Schwarz inequality, we get

$$|\nabla s|^2 \geq \frac{1}{n} |Ds|^2. \tag{35}$$

Combining (30), (33), (34), (35) and using (26), we have

$$\begin{aligned}
&\int_M |D_{\varphi}^* s|^2 e^{-\varphi} \\
&\geq \int_M \left[ \frac{1}{n} |D_{\varphi}^* s + \nabla \varphi \cdot s|^2 + \left( \frac{1}{2} \Delta \varphi - \frac{1}{4} |\nabla \varphi|^2 \right) |s|^2 \right] e^{-\varphi} \\
&\quad + \int_M \left( \lambda_{\mathbb{S}} |s|^2 + \frac{1}{4} |\nabla \varphi|^2 |s|^2 \right) e^{-\varphi} - \operatorname{Re} \int_M \left\langle \nabla \varphi \cdot s, D_{\varphi}^* s + \frac{1}{2} \nabla \varphi \cdot s \right\rangle e^{-\varphi} \\
&= \int_M \left[ \frac{1}{2} \Delta \varphi - \left( \frac{1}{2} - \frac{1}{n} \right) |\nabla \varphi|^2 + \lambda_{\mathbb{S}} \right] |s|^2 e^{-\varphi} \\
&\quad - 2 \left( \frac{1}{2} - \frac{1}{n} \right) \operatorname{Re} \int_M \langle \nabla \varphi \cdot s, D_{\varphi}^* s \rangle e^{-\varphi} + \frac{1}{n} \int_M |D_{\varphi}^* s|^2 e^{-\varphi} \tag{36} \\
&\geq \int_M \left[ \frac{1}{2} \Delta \varphi - \left( \frac{1}{2} - \frac{1}{n} \right) \left( 1 + \frac{1}{\varepsilon} \right) |\nabla \varphi|^2 + \lambda_{\mathbb{S}} \right] |s|^2 e^{-\varphi} \\
&\quad - \left( \left( \frac{1}{2} - \frac{1}{n} \right) \varepsilon - \frac{1}{n} \right) \int_M |D_{\varphi}^* s|^2 e^{-\varphi},
\end{aligned}$$

where  $\varepsilon$  is any positive constant. We have thus proved (28). The inequality (36) also gives the estimate (27). ■

To establish the  $L^2$ -existence theorem, we also need the following variant of Riesz representation Theorem:

**Lemma 2.6** (c.f. [15]) *Let  $T : H_1 \longrightarrow H_2$  be a closed and densely defined operator between Hilbert spaces. For any  $f \in H_2$  and any constant  $C > 0$ , the following conditions are equivalent*

1. *there exists some  $u \in \operatorname{Dom}(T)$  such that  $Tu = f$  and  $\|u\|_{H_1} \leq C$ .*
2.  *$|(f, s)_{H_2}| \leq C \|T^* s\|_{H_1}$  holds for any  $s \in \operatorname{Dom}(T^*)$ .*

We have the following proposition of the  $L^2$ -existence result for the Dirac operator.

**Proposition 2.7** *Let  $\mathbb{S}$  be a Dirac bundle over a smooth Riemannian manifold  $(M, g)$  and  $D$  be the Dirac operator. Let  $\varphi : M \rightarrow \mathbb{R}$  be a  $C^2$  function and  $\varepsilon > 0$  be a constant such that*

$$\Delta\varphi - \left(1 - \frac{2}{n}\right) \left(1 + \frac{1}{\varepsilon}\right) |\nabla\varphi|^2 + 2\lambda_{\mathbb{S}} \geq 0 \text{ on } M. \quad (37)$$

For any section  $f \in L^2_{\varphi}(M, \mathbb{S})$ , if

$$\int_M \frac{|f|^2}{\Delta\varphi - \left(1 - \frac{2}{n}\right) \left(1 + \frac{1}{\varepsilon}\right) |\nabla\varphi|^2 + 2\lambda_{\mathbb{S}}} e^{-\varphi} < \infty, \quad (38)$$

then there exists a section  $u \in L^2_{\varphi}(M, \mathbb{S})$  such that

$$Du = f \text{ and } \|u\|_{\varphi}^2 \leq \frac{1}{C} \int_M \frac{|f|^2}{\Delta\varphi - \left(1 - \frac{2}{n}\right) \left(1 + \frac{1}{\varepsilon}\right) |\nabla\varphi|^2 + 2\lambda_{\mathbb{S}}} e^{-\varphi},$$

where  $C = \frac{n}{2(n-1)+(n-2)\varepsilon}$ .

**Proof.** Fix some open subset  $\Omega$  which is relatively compact in  $M$ , we restrict  $\mathbb{S}$  to  $\Omega$  and introduce

$$H_1 = H_2 = L^2_{\varphi}(\Omega, \mathbb{S}).$$

We let the closed, densely defined operator

$$T : \text{Dom}(T) = \{u \in H_1 : Du \in H_2\} \longrightarrow H_2$$

to be the maximal differential operator defined by the Dirac operator (acting on smooth sections)  $D : \Gamma(\Omega, \mathbb{S}) \rightarrow \Gamma(\Omega, \mathbb{S})$ .

From Proposition 2.5 we have for any compact supported section  $s \in \Gamma(\Omega, \mathbb{S})$ ,

$$\int_M |D_{\varphi}^* s|^2 e^{-\varphi} \geq C \int_M \left( \Delta\varphi - \left(1 - \frac{2}{n}\right) \left(1 + \frac{1}{\varepsilon}\right) |\nabla\varphi|^2 + 2\lambda_{\mathbb{S}} \right) |s|^2 e^{-\varphi}. \quad (39)$$

Since  $T^*$  is given by the minimal differential operator defined by the formally adjoint operator  $D_{\varphi}^* : \Gamma(\Omega, \mathbb{S}) \rightarrow \Gamma(\Omega, \mathbb{S})$  (c.f. [16] for a more general result), we know that smooth sections of  $\mathbb{S}$  with compact support in  $\Omega$  are dense in  $\text{Dom}(T^*)$  with respect to the graph norm:

$$s \mapsto \sqrt{\|s\|_{H_2}^2 + \|T^* s\|_{H_1}^2}.$$

Because  $\varphi$  and  $\nabla\varphi$  are bounded on  $\Omega$ , (39) holds for any  $s \in \text{Dom}(T^*)$ .

Let  $f_{\Omega} \in L^2_{\varphi}(\Omega, \mathbb{S})$ . As we have proved that Proposition 2.5 is true for any  $s \in \text{Dom}(T^*)$ , by Cauchy-Schwarz inequality, we get the following estimate for any  $s \in \text{Dom}(T^*)$

$$|(f_{\Omega}, s)_{H_2}|^2 \leq \frac{1}{C} \int_{\Omega} \frac{|f_{\Omega}|^2}{\Delta\varphi - \left(1 - \frac{2}{n}\right) \left(1 + \frac{1}{\varepsilon}\right) |\nabla\varphi|^2 + 2\lambda_{\mathbb{S}}} e^{-\varphi} \cdot \|T^* s\|_{H_1}^2$$

Lemma 2.6 implies that there exists some  $u_\Omega \in \text{Dom}(T) \subseteq L_\varphi^2(\Omega, \mathbb{S})$  such that

$$Du_\Omega = f_\Omega \text{ and } \int_\Omega |u_\Omega|^2 e^{-\varphi} \leq \frac{1}{C} \int_\Omega \frac{|f_\Omega|^2}{\Delta\varphi - (1 - \frac{2}{n})(1 + \frac{1}{\varepsilon})|\nabla\varphi|^2 + 2\lambda_\mathbb{S}} e^{-\varphi}. \quad (40)$$

Given  $f \in L_\varphi^2(M, \mathbb{S})$  with  $\int_M \frac{|f|^2}{\Delta\varphi - (1 - \frac{2}{n})(1 + \frac{1}{\varepsilon})|\nabla\varphi|^2 + 2\lambda_\mathbb{S}} e^{-\varphi} < \infty$ , we can finish the proof by applying (40) to an increasing sequence of open subsets  $\Omega_0 \subset\subset \Omega_1 \subset\subset \cdots \nearrow M$  to get a sequence of  $u_\nu \in L_\varphi^2(\Omega_\nu, \mathbb{S})$  such that  $Du_\nu = f|_{\Omega_\nu}$  in the sense of distribution and

$$\int_{\Omega_\nu} |u_\nu|^2 e^{-\varphi} \leq \frac{1}{C} \int_{\Omega_\nu} \frac{|f|^2}{\Delta\varphi - (1 - \frac{2}{n})(1 + \frac{1}{\varepsilon})|\nabla\varphi|^2 + 2\lambda_\mathbb{S}} e^{-\varphi}, \nu = 0, 1, \dots$$

The above uniform  $L^2$ -estimate allows us obtain a desired solution of  $Du = f$  by taking a weak limit of  $\{u_\nu\}$ . The proof is complete. ■

**Remark 2.8** *Different from the Bochner formula of the  $\bar{\partial}$ -operator, for the Dirac operator  $D$ , only  $|Ds|^2$  is involved on the left side of (25). This simplifies the  $L^2$ -estimates in our setting compared to [15], as we only need the minimal extension of the Dirac operator  $D$ , for which the set of compactly supported smooth sections is dense in its domain with respect to the graph norm.*

### 3 The first eigenvalue of Dirac operators

By constructing the weight function  $\varphi$  in Theorem 1.1 from certain Poisson equation, we obtain the following

**Corollary 3.1** *If  $M$  is a closed 2-dimensional Riemannian manifold, then*

$$\lambda_{\min}(D^2) \geq \frac{2}{\text{Vol}(M)} \int_M \lambda_\mathbb{S}, \quad (41)$$

where  $\lambda_{\min}(D^2)$  is the first eigenvalue of  $D^2$ . Consequently, if

$$\int_M \lambda_\mathbb{S} > 0, \quad (42)$$

then  $D : L^2(M, \mathbb{S}) \rightarrow L^2(M, \mathbb{S})$  defines an isomorphism.

**Proof.** Since  $\lambda_{\min}(D^2) \geq 0$ , the inequality (41) is trivial if  $\int_M \lambda_\mathbb{S} \leq 0$ . Now let us assume  $\int_M \lambda_\mathbb{S} > 0$ . The equation  $\Delta\varphi + 2\lambda_\mathbb{S} = \frac{2}{\text{Vol}(M)} \int_M \lambda_\mathbb{S}$  always has a solution  $\varphi$ , as the integral of  $\lambda_\mathbb{S} - \frac{1}{\text{Vol}(M)} \int_M \lambda_\mathbb{S}$  over  $M$  is zero. (In fact,

$\Delta\varphi + 2\lambda_{\mathbb{S}} > 0$  has a solution  $\varphi$  if and only if  $\int_M \lambda_{\mathbb{S}} > 0$ ). Take such  $\varphi$  as the weight function in (2). For any  $f \in L^2(M, \mathbb{S})$ , by our condition

$$\int_M \frac{|f|^2}{\Delta\varphi + 2\lambda_{\mathbb{S}}} e^{-\varphi} = \frac{\text{Vol}(M)}{2 \int_M \lambda_{\mathbb{S}}} \int_M |f|^2 e^{-\varphi} < \infty.$$

By Theorem 1.1, there is a  $L^2$  section  $u$  such that  $Du = f$ . From (2) we have

$$\frac{\int_M |Du|^2 e^{-\varphi}}{\int_M |u|^2 e^{-\varphi}} \geq \frac{2}{\text{Vol}(M)} \int_M \lambda_{\mathbb{S}}. \quad (43)$$

By the Rayleigh quotient, the desired estimate  $\lambda^2 \geq \frac{2}{\text{Vol}(M)} \int_M \lambda_{\mathbb{S}}$  follows, where  $\lambda$  is a square root of  $\lambda_{\min}(D^2)$ . (To get rid of the weight  $e^{-\varphi}$  in (43), we consider the operator  $\tilde{D} = e^{-\frac{\varphi}{2}} \circ D \circ e^{\frac{\varphi}{2}}$  and section  $\tilde{u} = e^{-\frac{\varphi}{2}} u$ , and notice  $D^2$  and  $\tilde{D}^2$  have the same eigenvalues). ■

**Remark 3.2** *Corollary 3.1 also follows from (29), by taking the same  $\varphi$  as above and the nontrivial section  $s$  such that  $D(e^{-\varphi}s) = \lambda e^{-\varphi}s$ . A different proof was given in Theorem 2 of [2].*

**Proposition 3.3** *If  $M$  is a noncompact 2-dimensional Riemannian manifold, then for any  $f \in L^2_{loc}(M, \mathbb{S})$ , there exists a section  $u \in L^2_{loc}(M, \mathbb{S})$  such that  $Du = f$ , where  $L^2_{loc}(M, \mathbb{S})$  is the space of locally square integrable sections of  $\mathbb{S}$ .*

**Proof.** By Theorem 1.1, it suffices to prove that there is a  $\varphi \in C^2(M)$  such that

$$\Delta\varphi + 2\lambda_{\mathbb{S}} \geq 0 \text{ on } M \text{ and } \int_M \frac{|f|^2}{\Delta\varphi + 2\lambda_{\mathbb{S}}} e^{-\varphi} < \infty. \quad (44)$$

We first construct some nonnegative proper exhaustion function  $\psi \in C^2(M)$  such that  $\Delta\psi + 2\lambda_{\mathbb{S}} \geq 1$  on  $M$ . Since  $M$  is a noncompact 2-dimensional Riemannian manifold, there always exists a nonnegative exhaustion function  $\phi \in C^\infty(M)$  which is strictly subharmonic. Then, we choose a nonnegative function  $\kappa \in C^\infty[0, +\infty)$  such that

$$\kappa'(t) > 0, \kappa''(t) \geq 0 \text{ for } t \geq 0, \kappa'(\nu) \geq \sup_{\Omega_{\nu+1} \setminus \Omega_\nu} \frac{1 - 2\lambda_{\mathbb{S}}}{\Delta\phi} \text{ for } \nu = 0, 1, 2, \dots, \quad (45)$$

where  $\Omega_\nu := \{x \in M \mid \phi(x) < \nu\}$  ( $\nu = 0, 1, 2, \dots$ ). Set  $\psi = \kappa \circ \phi$ , then

$$\Delta\psi = \kappa' \circ \phi \cdot \Delta\phi + \kappa'' \circ \phi \cdot |\nabla\phi|^2 \geq \kappa' \circ \phi \cdot \Delta\phi.$$

Consequently, by the monotonicity of  $\kappa'$  and (45), we obtain  $\Delta\psi + 2\lambda_{\mathbb{S}} \geq 1$  on  $M$ . Since  $\kappa(t) \rightarrow +\infty$  as  $t \rightarrow +\infty$ ,  $\psi = \kappa \circ \phi$  is also an exhaustion function.

Now we construct the desired function  $\varphi$  satisfying (44). If we set  $\Omega_\nu = \psi^{-1}(-\infty, \nu)$ ,  $\nu = 0, 1, 2, \dots$ , then  $\emptyset = \Omega_0 \subset \subset \Omega_1 \subset \subset \Omega_2 \subset \subset \dots \nearrow M$ . Let  $\chi \in C^\infty[0, +\infty)$  such that

$$\chi(\nu) \geq \nu + \log \int_{\Omega_{\nu+1}} |f|^2 \quad (\nu = 0, 1, 2, \dots), \quad \chi' \geq 1, \quad \chi'' \geq 0.$$

Define  $\varphi = \chi \circ \psi$ , then we have

$$\Delta\varphi = \chi' \circ \psi \cdot \Delta\psi + \chi'' \circ \psi \cdot |\nabla\psi|^2 \geq \Delta\psi \geq 1 - 2\lambda_{\mathbb{S}},$$

and

$$\begin{aligned} \int_M \frac{|f|^2}{\Delta\varphi + 2\lambda_{\mathbb{S}}} e^{-\varphi} &= \sum_{\nu \geq 0} \int_{\Omega_{\nu+1} \setminus \Omega_\nu} \frac{|f|^2}{\Delta\varphi + 2\lambda_{\mathbb{S}}} e^{-\varphi} \\ &\leq \sum_{\nu \geq 0} e^{-\chi(\nu)} \int_{\Omega_{\nu+1} \setminus \Omega_\nu} |f|^2 \\ &\leq \sum_{\nu \geq 0} e^{-\nu} < \infty. \end{aligned}$$

Hence, we have constructed the desired the weight function  $\varphi$  satisfying (44) and the proof is therefore complete. ■

**Corollary 3.4** *The Poisson equation  $\Delta_d u = f$  is always solvable on noncompact 2-dimensional Riemannian manifolds (we do not require  $M$  is orientable).*

**Proof.** For any 2-dimensional Riemannian manifold  $(M, g)$ , the vector bundle

$\Lambda^* M := \bigoplus_{\ell=0}^2 \Lambda^\ell(T^* M)$  has a natural structure of a Dirac bundle over  $(M, g)$ , and

the associated Dirac operator is given by  $D = d + \delta$ , where  $\delta$  is the codifferential operator. Since  $D^2 = (d + \delta)^2 = \Delta_d$  (the Hodge Laplacian operator), by the above proposition, the corollary follows. ■

By a suitable choice of the weight function in our estimate (27), we can give a simple proof of Bär's

**Theorem 3.5** *(Theorem 3 [2]) Let  $\mathbb{S}$  be a Dirac bundle over a compact  $n$ -dimensional Riemannian manifold  $(M, g)$  without boundary, and  $D$  be the Dirac operator,  $n \geq 2$ . Then*

$$\lambda_{\min}(D^2) \geq \frac{n}{n-1} \lambda_{\min}(\mathbb{L})$$

where  $\lambda_{\min}(\cdot)$  means the first eigenvalue,  $\mathbb{L} = -\frac{n-1}{n-2} \Delta + \lambda_{\mathbb{S}}$  if  $n \geq 3$ , and  $\mathbb{L} = -\frac{1}{2} \Delta + \lambda_{\mathbb{S}}$  if  $n = 2$ .

**Proof.** For any  $\varphi \in C^\infty(M)$ , we can choose a non-trivial section  $s \in \Gamma(M, \mathbb{S})$  such that

$$D(e^{-a\varphi}s) = \lambda e^{-a\varphi}s$$

where  $\lambda$  is a square root of  $\lambda_{\min}(D^2)$  and  $a = 0$  if  $n = 2$ ,  $a = \frac{n}{2(n-1)}$  if  $n \geq 3$ . By Lemma 2.4, we know

$$D_\varphi^*s = D_{a\varphi}^*s + (a-1)\nabla\varphi \cdot s = \lambda s + (a-1)\nabla\varphi \cdot s.$$

Substituting the above identity into (27) and noticing  $\operatorname{Re}\langle s, \nabla\varphi \cdot s \rangle = 0$ , we have

$$\lambda^2 \int_M |s|^2 e^{-\varphi} \geq \int_M (\Delta\varphi - |\nabla\varphi|^2 + 2\lambda_{\mathbb{S}}) |s|^2 e^{-\varphi}, \text{ if } n = 2, \quad (46)$$

and

$$\lambda^2 \int_M |s|^2 e^{-\varphi} \geq \frac{n}{n-1} \int_M \left( \frac{1}{2}\Delta\varphi - \frac{n-2}{4(n-1)}|\nabla\varphi|^2 + \lambda_{\mathbb{S}} \right) |s|^2 e^{-\varphi}, \text{ if } n \geq 3. \quad (47)$$

Let  $\psi \in C^\infty(M)$  be an eigenfunction

$$\mathbb{L}\psi = \lambda_{\min}(\mathbb{L})\psi.$$

Without loss of generality, we may assume  $\psi > 0$  on  $M$ . Set

$$\varphi = -\log \psi \text{ if } n = 2, \text{ and } \varphi = -\frac{2(n-1)}{n-2} \log \psi \text{ if } n \geq 3,$$

then Theorem 3.5 follows from (46) and (47). ■

An example of a Dirac bundle over a 3-manifold is the normal bundle of an instanton in a  $G_2$  manifold. In Physics,  $G_2$ -manifolds are internal spaces for compactification in M-theory in eleven dimensional spacetimes, similar to the role of Calabi-Yau threefolds in string theory. Counting instantons in  $G_2$  manifolds is similar to counting holomorphic curves in Calabi-Yau threefolds.

**Definition 3.6** *A  $G_2$  manifold is a 7-dimensional Riemannian manifold  $(M, g)$  equipped with a parallel cross product  $\times$ . An instanton (or associative submanifold)  $A$  is a 3-dimensional submanifold whose tangent spaces are closed under the cross product.*

Let  $N_{A/M}$  be the normal bundle of  $A$  in  $M$ . Regarding  $N_{A/M}$  as a left Clifford module over  $A$  with the  $G_2$  cross product  $\times$  as the Clifford multiplication, it is a *twisted spinor bundle* over  $A$ , with the normal connection  $\nabla^\perp$  inherited from the Levi-Civita connection  $\nabla$  on  $M$  (Section 5 [22]). All 3-manifolds nearby  $A$  can be parameterized by sections  $V$  of  $N_{A/M}$ .

Given the cross product  $\times$ , one can define a  $TM$ -valued 3-form  $\tau$  on  $M$  as

$$\tau(u, v, w) = -u \times (v \times w) - g(u, v)w + g(u, w)v,$$

for  $u, v$  and  $w \in TM$ . Then  $A$  is an instanton if and only if  $\tau|_A = 0$  (c.f. [13]). Using  $\tau$  McLean (Section 5 [22]) defined a nonlinear function

$$F : C^{1,\alpha}(A, N_{A/M}) \rightarrow C^\alpha(A, N_{A/M}) \quad (0 < \alpha < 1)$$

such that instantons nearby  $A$  correspond to the zeros of  $F$  (the choice of the *Schauder* functional analysis setting over the  $W^{1,p}$  setting is necessary, due to the *cubic nonlinearity* of  $\tau$  and  $F$ ). He computed

$$\left. \frac{d}{dt} \right|_{t=0} F(tV) = DV, \quad (48)$$

where  $V \in A, N_{A/M}$ , and  $D$  is the *twisted Dirac operator* on  $N_{A/M}$  over  $A$ . Then he proved

**Theorem 3.7** (*Theorem 5-2 [22]*) *Infinitesimal deformations of instantons at  $A$  are parametrized by the space of harmonic twisted spinors on  $A$ , i.e. the kernel of  $D$ .*

More detailed exposition of McLean's proof can be found in Theorem 9 and Section 4.2 of [20].

We relate Theorem 3.5 to *rigidity* of instantons in  $G_2$  manifolds, i.e. situation that the moduli space of instantons near  $A$  is a zero dimensional smooth manifold.

**Corollary 3.8** *If an instanton  $A$  is compact and  $\lambda_{\min}(\mathbb{L}) > 0$ , then  $A$  is rigid. Here  $\mathbb{L} = -2\Delta_{(A,g)} + \lambda_{N_{A/M}}$ , and  $\Delta_{(A,g)}$  is the Laplace-Beltrami operator on  $A$  with the induced metric from  $(M, g)$ .*

**Proof.** Let  $\mathbb{S}$  be  $N_{A/M}$  and  $D$  be the twisted Dirac operator in Theorem 3.5. By our condition the kernel and cokernel of  $D : W^{1,2}(A, N_{A/M}) \rightarrow L^2(A, N_{A/M})$  vanish (note  $D$  is self-adjoint). A standard interpolation argument implies that  $D : W^{1,p}(A, N_{A/M}) \rightarrow L^p(A, N_{A/M})$  ( $p > 3$ ) is surjective (e.g. estimates between (36)~(37) in [20]), which in turn implies that  $D : C^{1,\alpha}(A, N_{A/M}) \rightarrow C^\alpha(A, N_{A/M})$  is surjective by the Schauder estimate of  $D$  and Sobolev embedding  $W^{1,p} \hookrightarrow C^0$ . By the implicit function theorem for the nonlinear function  $F$ , the moduli space  $F^{-1}(0)$  near  $A$  is a zero dimensional smooth manifold, and  $A$  is rigid. ■

Clearly if  $\lambda_{N_{A/M}} > 0$  everywhere on  $M$ , then  $\lambda_{\min}(\mathbb{L}) > 0$ , but not vice versa. So Corollary 3.8 provides a potentially weaker condition to guarantee the rigidity of  $A$ . The computation of  $\mathfrak{R}_{N_{A/M}}$  in terms of the curvature of  $M$

and the second fundamental form of  $A$  can be found in Section 5.3 of [6].



## 4 $\mathbb{Z}_2$ -graded Dirac operators

The solvability of the “half” Dirac equation  $D^\pm u = f$  is of interest for several reasons: e.g. in quaternionic analysis, the correct generalization of analytical functions on  $\mathbb{C}$  are solutions of  $D^+ u = 0$  on  $\mathbb{H}$ , as requiring  $u$  to be quaternion differentiable only yields linear functions;  $D^\pm$  may have nonzero Fredholm index to produce nontrivial invariants;  $D^\pm$  arises as the linearized operator (modulo zeroth order terms) in many moduli problems, like those of  $J$ -holomorphic curves and solutions of the Seiberg-Witten equation. As we will see, consideration of the  $\mathbb{Z}_2$ -grading also improves eigenvalue estimates of the Dirac operator  $D$  on even dimensional manifolds.

Our main technical tool, Proposition 2.5, extends immediately to  $D^\pm$ :

**Proposition 4.1** *For any smooth section  $s$  of  $\mathbb{S}^\mp$  with compact support and any  $C^2$  function  $\varphi : M \rightarrow \mathbb{R}$ , we have*

$$\begin{aligned} & \frac{n-1}{n} \int_M |(D^\pm)_\varphi^* s|^2 e^{-\varphi} + \frac{n-2}{n} \operatorname{Re} \int_M \langle \nabla \varphi \cdot s, (D^\pm)_\varphi^* s \rangle e^{-\varphi} \\ & \geq \int_M \left[ \frac{1}{2} \Delta \varphi - \left( \frac{1}{2} - \frac{1}{n} \right) |\nabla \varphi|^2 + \lambda_{\mathbb{S}^\mp} \right] |s|^2 e^{-\varphi}. \end{aligned}$$

and

$$\int_M |(D^\pm)_\varphi^* s|^2 e^{-\varphi} \geq C \int_M \left[ \Delta \varphi - \left( 1 - \frac{2}{n} \right) \left( 1 + \frac{1}{\varepsilon} \right) |\nabla \varphi|^2 + 2\lambda_{\mathbb{S}^\mp} \right] |s|^2 e^{-\varphi}.$$

where  $C$  is the constant in Proposition 2.5.

**Proof.** Consider the Dirac operator  $D^+ : L_\varphi^2(M, \mathbb{S}^+) \rightarrow L_\varphi^2(M, \mathbb{S}^-)$  (the  $D^-$  case is similar). Let  $(D^+)_\varphi^*$  be its formal adjoint with respect to the measure  $e^{-\varphi} d\operatorname{vol}_g$ . Then it is easy to observe

$$(D^+)_\varphi^* s = e^\varphi D^- (e^{-\varphi} s) = e^\varphi D (e^{-\varphi} s) = -\nabla \varphi \cdot s + D^- s \quad (49)$$

for smooth sections  $s$  of  $\mathbb{S}^-$ , where we have used that  $D^- s = D|_{\mathbb{S}^-}(s)$ .

When we restrict the sections from  $\Gamma(\mathbb{S})$  to  $\Gamma(\mathbb{S}^-)$ , by (20) and (21),  $D^2$  becomes  $D^+ D^-$ , and  $\mathfrak{R}$  becomes  $\mathfrak{R}^-$  in the Bochner formula (17), i.e.

$$D^+ D^- = \nabla^* \nabla + \mathfrak{R}^-.$$

Similarly

$$D^- D^+ = \nabla^* \nabla + \mathfrak{R}^+.$$

Integrating on  $M$  we obtain

$$\int_M |D^\pm s|^2 = \int_M |\nabla s|^2 + \int_M \langle \mathfrak{R}^\pm s, s \rangle. \quad (50)$$

The remaining part of the proof is the same as Proposition 2.5, except that  $\lambda_{\mathbb{S}}$  is replaced by  $\lambda_{\mathbb{S}^-}$ , where

$$\lambda_{\mathbb{S}^\pm}(x) := \text{the smallest eigenvalue of } \mathfrak{R}^\pm(x) \quad (51)$$

at any  $x \in M$ . The proposition follows.  $\blacksquare$

**Remark 4.2** From Proposition 4.1, we know that Proposition 2.7 still holds if we replace simultaneously  $D$  by  $D^\pm$  and  $\lambda_{\mathbb{S}}$  by  $\lambda_{\mathbb{S}^\mp}$ .

From Proposition 4.1 and Remark 4.2, similarly we obtain the results of  $D^\pm$  parallel to Theorem 1.1, Corollary 3.1 and Theorem 1.6, by replacing  $D$  by  $D^\pm$ , and  $\lambda_{\mathbb{S}}$  by  $\lambda_{\mathbb{S}^\mp}$  in the corresponding statements. For example, we can refine Corollary 3.1 to Corollary 1.2. From this we obtain Corollary 1.3 as follows.

**Proof.** (of Corollary 1.3). Let  $\mathbb{S}^+ = E$  and  $\mathbb{S}^- = \wedge^{0,1}(E)$  over the Riemann surface  $M$ ,  $\bar{\partial} : E \rightarrow \wedge^{0,1}(E)$  be the Cauchy-Riemann operator, and  $\bar{\partial}^*$  be its adjoint. Let  $D^+ = \sqrt{2}\bar{\partial}$ ,  $D^- = \sqrt{2}\bar{\partial}^*$ , then  $D = (D^+, D^-)$  is the Dolbeault-Dirac operator on  $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ . From Lemma 2.2, we have the curvature operators  $\mathfrak{R}^\pm$  of  $\mathbb{S}^\pm$  as

$$\mathfrak{R}^- = \sqrt{-1}\Lambda R^E + K, \quad \mathfrak{R}^+ = -\sqrt{-1}\Lambda R^E.$$

By definitions (22) and (23), we have

$$\begin{aligned} \lambda_{\mathbb{S}^+} &= \text{the smallest eigenvalue of } -\sqrt{-1}\Lambda R^E = -\Theta_E, \\ \lambda_{\mathbb{S}^-} &= \text{the smallest eigenvalue of } \sqrt{-1}\Lambda R^E + K = \theta_E + K. \end{aligned}$$

Applying Corollary 1.2 to  $D$ , we obtain Corollary 1.3.  $\blacksquare$

By Corollary 1.2, we improve the estimate for  $\lambda_{\min}(D^2)$  in Corollary 3.1 as follows.

**Corollary 4.3** Under the hypothesis in Corollary 1.2, it holds that

$$\lambda_{\min}(D^2) \geq \frac{2}{\text{Vol}(M)} \min \left\{ \int_M \lambda_{\mathbb{S}^+}, \int_M \lambda_{\mathbb{S}^-} \right\}.$$

From Proposition 4.1, we can also improve Theorem 3.5 of Bär. As in the proof of Theorem 3.5, let  $s \in \Gamma(M, \mathbb{S})$  be a non-trivial section such that  $D(e^{-a\varphi}s) = \lambda e^{-a\varphi}s$ , which implies

$$(D^\pm)^*_{a\varphi} s^\mp = \lambda s^\pm.$$

By applying Proposition 4.1 to  $s^\mp$  in the same way as in the proof of Theorem 3.5, we have the following

**Corollary 4.4** Let  $\mathbb{S}$  be a  $\mathbb{Z}_2$ -graded Dirac bundle over a compact  $n$ -dimensional Riemannian manifold  $(M, g)$  without boundary, and  $D$  be the Dirac operator,  $n \geq 2$ . Then

$$\lambda_{\min}(D^2) \geq \frac{n}{n-1} \min \{ \lambda_{\min}(\mathbb{L}^+), \lambda_{\min}(\mathbb{L}^-) \}$$

where  $\mathbb{L}^\pm = -\frac{n-1}{n-2}\Delta + \lambda_{\mathbb{S}^\pm}$  if  $n \geq 3$ , and  $\mathbb{L}^\pm = -\frac{1}{2}\Delta + \lambda_{\mathbb{S}^\pm}$  if  $n = 2$ .

## 5 Manifolds with cylindrical ends

Between the compact and noncompact cases, there is an important case of manifolds with cylindrical ends. There are many works of differential operators on such manifolds, going back to the work of Lockhart and McOwen ([21]), and occurring often in gauge theory and low dimensional topology (e.g. [5], [27]).

**Definition 5.1** *Let  $(M, g)$  be a  $n$ -dimensional Riemannian manifold. We say it is a manifold with cylindrical ends if outside a compact subset,  $M$  consists of product Riemannian manifolds  $E_\nu \simeq [0, +\infty) \times B_\nu$  ( $\nu = 1, 2, \dots, m$ ), where each  $B_\nu$  is a  $(n-1)$ -dimensional compact Riemannian manifold. Each  $E_\nu$  is called a cylindrical end, and the  $[0, +\infty)$  direction is called the cylindrical direction.*

For analysis on cylindrical manifolds, one often needs the *Sobolev space with exponential weights*. In other words, one chooses a smooth weight function  $\varphi$  on  $M$  such that

$$\varphi|_{E_\nu} = -\delta_\nu \tau_\nu \text{ for some constant } \delta_\nu \geq 0, \text{ and large } \tau_\nu \quad (52)$$

( $\nu = 1, 2, \dots, m$ ), and then defines the *weighted Sobolev norm*

$$\|f\|_{W_\delta^{k,p}} := \left\| e^{-\frac{\varphi}{p}} f \right\|_{W^{k,p}}, \quad (53)$$

where  $\delta = (\delta_1, \dots, \delta_m)$ . Different choices of  $\varphi$  satisfying (52) only result in equivalent Banach spaces.

**Proof.** (of Theorem 1.6). Without loss of generality we assume  $M \setminus K = \cup_{\nu=1}^m E_\nu$ , where  $E_\nu$  are the cylindrical ends of  $M$ , (otherwise we can always enlarge  $K$ ). We choose a smooth cut-off function  $\rho : M \rightarrow [0, 1]$  such that  $\rho \equiv 0$  on  $K$ , and on  $E_\nu$   $\rho$  is a function of the variable  $\tau_\nu$  in the cylindrical direction satisfying

$$\rho(\tau_\nu) = \begin{cases} 0, & \text{if } 0 \leq \tau_\nu \leq 1 \\ 1, & \text{if } \tau_\nu \geq 2 \end{cases}, \text{ and } 0 \leq \rho'(\tau_\nu), |\rho''(\tau_\nu)| \leq 2.$$

Let  $\mu > 0$  be the first eigenvalue of the Dirichlet eigenvalue problem

$$-\Delta \eta = \mu \eta \text{ on } M \setminus \cup_{\nu=1}^m (3, \infty) \times B_\nu. \quad (54)$$

By the nodal domain theorem, we can find an eigenfunction function  $\eta > 0$  on  $M \setminus \cup_{\nu=1}^m (3, \infty) \times B_\nu$ . Setting  $A = \frac{1}{(1-\frac{2}{n})(1+\frac{1}{\varepsilon})} > 0$ , from (54) it follows that for sufficiently large  $\varepsilon > 0$

$$\begin{aligned} \left( -\frac{1}{(1-\frac{2}{n})(1+\frac{1}{\varepsilon})} \Delta + 2\lambda_\mathbb{S} \right) \eta^\gamma &= \eta^{\gamma-2} \left[ -A\gamma\eta\Delta\eta - A\gamma(\gamma-1)|\nabla\eta|^2 + 2\lambda_\mathbb{S}\eta^2 \right] \\ &\geq \eta^{\gamma-1} [-A\gamma\Delta\eta + 2\lambda_\mathbb{S}\eta] \\ &\geq \eta^\gamma [A\gamma\mu - 2\beta] > 0 \end{aligned} \quad (55)$$

on  $M \setminus \cup_{\nu=1}^m [2, \infty) \times B_\nu$ , provided that  $\beta$  satisfies the condition

$$0 < \beta < \frac{\gamma\mu}{2 - \frac{4}{n}},$$

where  $\gamma \in (0, 1)$  is a constant to be determined. Let

$$\phi := -\frac{1}{\left(1 - \frac{2}{n}\right)\left(1 + \frac{1}{\varepsilon}\right)} \log \eta,$$

then on  $M \setminus \cup_{\nu=1}^m [2, \infty) \times B_\nu$ , by (55) we have

$$\begin{aligned} & \Delta(\gamma\phi) - \left(1 - \frac{2}{n}\right) \left(1 + \frac{1}{\varepsilon}\right) |\nabla(\gamma\phi)|^2 + 2\lambda_{\mathbb{S}} \\ &= \eta^{-\gamma} \left[ -\frac{1}{\left(1 - \frac{2}{n}\right)\left(1 + \frac{1}{\varepsilon}\right)} \Delta\eta^\gamma + 2\lambda_{\mathbb{S}}\eta^\gamma \right] > 0. \end{aligned} \quad (56)$$

We define a function  $h : \cup_{\nu=1}^m E_\nu \rightarrow \mathbb{R}$ , such that

$$h(\tau_\nu, b_\nu) = -\delta_\nu \tau_\nu, \quad (57)$$

where the constants  $\delta_\nu \geq 0$  are to be determined. Then we define the weight function  $\varphi : M \rightarrow \mathbb{R}$  as

$$\varphi = \gamma(1 - \rho)\phi + \rho h. \quad (58)$$

It is easy to check that  $\varphi$  is smooth and is globally defined on  $M$ .

By (56)  $\sim$  (58), for sufficiently small  $\delta_1, \dots, \delta_m \geq 0$  and sufficiently large  $\varepsilon > 0$ , we have

$$\begin{cases} > 0 & \text{on } K \cup \cup_{\nu=1}^m [0, 1] \times B_\nu, \\ = -\left(1 - \frac{2}{n}\right) \left(1 + \frac{1}{\varepsilon}\right) \delta_\nu^2 + 2\lambda_{\mathbb{S}} > 0 & \text{on } \cup_{\nu=1}^m [2, \infty) \times B_\nu. \end{cases}$$

On each  $[1, 2] \times B_\nu$ , if  $\gamma$  and  $\delta_1, \dots, \delta_m$  are sufficiently small, then

$$\begin{aligned} & \Delta\varphi - \left(1 - \frac{2}{n}\right) \left(1 + \frac{1}{\varepsilon}\right) |\nabla\varphi|^2 + 2\lambda_{\mathbb{S}} \\ & \geq -C_1(\gamma + |h| + |\nabla h|) + 2\lambda_{\mathbb{S}} \geq \alpha, \end{aligned} \quad (59)$$

where  $C_1 > 0$  is some constant depending on  $\phi$  and  $\rho$ . So we have

$$\Delta\varphi - \left(1 - \frac{2}{n}\right) \left(1 + \frac{1}{\varepsilon}\right) |\nabla\phi|^2 + 2\lambda_{\mathbb{S}} \geq \alpha_1 \text{ on } M \quad (60)$$

for some constant  $\alpha_1 > 0$ .

Therefore by Proposition 2.7 and (60), for any  $f \in L_\varphi^2(M, \mathbb{S})$ , there exists  $u \in L_\varphi^2(M, \mathbb{S})$  such that  $Du = f$  and

$$\|u\|_\varphi^2 \leq C_2 \int_M \frac{|f|^2}{\Delta\varphi - \left(1 - \frac{2}{n}\right) \left(1 + \frac{1}{\varepsilon}\right) |\nabla\phi|^2 + 2\lambda_{\mathbb{S}}} e^{-\varphi} \leq C \int_M |f|^2 e^{-\varphi}$$

for some constants  $C_2$  and  $C$ . By elliptic regularity of  $D$  and the cylindrical structure on  $M$ , we have  $\|u\|_\varphi + \|\nabla u\|_\varphi \leq C \|f\|_\varphi$ , so (11) is proved.

With the  $L^2$ -estimate (11) ( $\delta = 0$  case) on the cylindrical manifold  $M$ , it is standard to derive the  $L^p$  estimate (12) (c.f. Section 3.4 [5]). The theorem is proved. ■

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